Studies on the Effect of Beam-Coupling Impedance on Schottky Spectra of Bunched Beams

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I. Introduction

Schottky monitors can be used for **non-invasive beam diagnostics** to estimate various bunch characteristics, such as tune, chromaticity, bunch profile or synchrotron frequency distribution. However, collective effects, in particular **beam-coupling impedance, can significantly affect Schottky spectra** when large bunch charges are involved. In such conditions, the available interpretation methods are difficult to apply directly to the measured spectra, thus **preventing the extraction of beam and machine parameters**, which is possible for lower bunch charges.

To study the impact of impedance on such spectra, we perform here time-domain, macro-particle simulations and apply a semi-analytical method to compute the Schottky signal for various machine and beam conditions, including those corresponding to typical physics operation at the Large Hadron Collider (LHC). This study provides preliminary interpretations of how the Schottky spectra are affected by a longitudinal broad-band resonator (both theoretically and through simulations) and by a transverse broad-band resonator (through simulation).

II. Theoretical description (longitudinal dynamics)

Theoretical reconstructions of Schottky spectra, such as the matrix formalism [1] or the Monte Carlo approach [2-4] **assume that the synchrotron frequency distribution is known**. When the particles are moving freely in the potential well of the radio frequency (RF) bucket, an analytical relation between the amplitude of the synchrotron oscillation and its frequency can be used, allowing these methods to reconstruct the Schottky spectrum from the synchrotron amplitude distribution. However, **this relation has to be modified when beam-coupling impedance affects the longitudinal dynamics.**

Synchrotron oscillation without external forces

The **equation of motion** for the RF phase ϕ of a given particle is:

$$\frac{d^2\phi}{dt^2} + \Omega_0^2 \sin\phi = 0$$

(non-harmonic motion)

Exact oscillation frequency

$$\Omega_s(\hat{\phi}) = \frac{\pi}{2\mathcal{K}\left[\sin\left(\frac{\hat{\phi}}{2}\right)\right]}\Omega_0$$

(K: complete elliptic integral of the first kind)

With $\boldsymbol{\Omega_0}$ **the nominal synchrotron frequency** (i.e. the limit synchrotron frequency for **synchrotron amplitude** $\hat{\phi}$ approaching zero).

This equation is **identical to** the one of the **non-linear physical pendulum** and **solutions and approximations of this equation exist** in the literature [5].

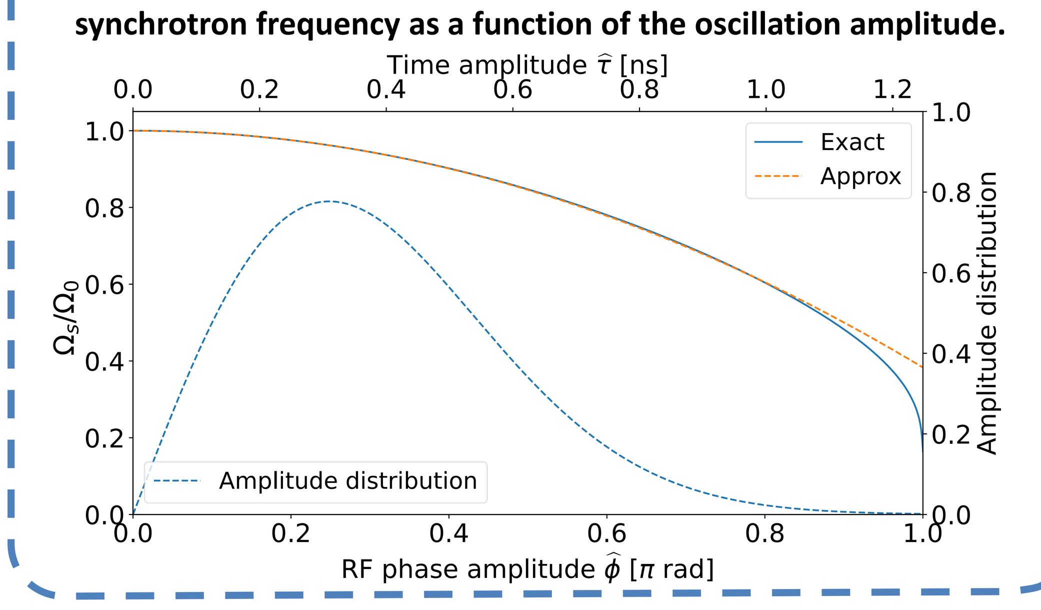
Third order approximation

$$\frac{d^2\phi}{dt^2} + \Omega_0^2\left(\phi - \frac{\phi^3}{6}\right) + O(\phi^5) = 0$$

Approx. osc. frequency

$$\Omega_s(\hat{\phi}) = \Omega_0\left(1 - \frac{\hat{\phi}^2}{16}\right)$$

Comparison between the exact and approximate expressions of the synchrotron frequency as a function of the oscillation amplitude.



Equation of motion with impedance

Additional external forces, such as the one coming from **beam-coupling impedance**, will influence the longitudinal dynamics of the particle [6].

➔ With additional forces, the previous equation of motion becomes:

$$\ddot{\phi} + \Omega_0^2 \sin\phi = \frac{\eta h \omega_0}{p_0}\sum_i F_i(t)$$

Beam-coupling impedance

$$F_{Imp}(t) = e\left[\vec{E} + \vec{\beta}c \times \vec{B}\right]_{\parallel}\left(t, z = c\tau(t)\right) = \frac{-Ie}{C}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p)e^{jp\omega_0\tau(t)}$$

With the notations:

- η : slippage factor.
- p_0 : reference momentum.
- h : RF harmonic number.
- $\hat{\lambda}(p) \coloneqq \hat{\lambda}(p\omega_0)$: bunch spectrum.
- C : accelerator circumference.
- $Z_{\parallel}(p) \coloneqq Z_{\parallel}(p\omega_0)$: longitudinal impedance.
- e : elementary charge.
- $I = Ne/T_0$: bunch current.
- τ : time arrival difference between a given particle and the synchronous particle.

$$\ddot{\phi} + \Omega_0^2 \overbrace{\sin\phi}^{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\phi^{2n+1}} = \Omega_0^2 \frac{I}{\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p)e^{\overbrace{j\frac{p}{h}\phi}^{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{jp}{h}\right)^n\phi^n}}$$

Expanding the sine and exponential function with their Maclaurin series.

The idea is that, for small oscillation amplitudes, only the first order terms can be kept, while for larger amplitudes, higher order terms can be taken into account.

$$\text{(1)} \quad \boxed{\ddot{\phi} + \Omega_0^2\sum_{n=0}^{\infty} S_n\phi^n = 0}$$

General equation of motion **with impedance**

With the coefficients:

$$S_n = \begin{cases} -Z_n & : n \text{ even} \\ \frac{j^{n-1}}{n!} - Z_n & : n \text{ odd} \end{cases}$$

$$Z_n = \frac{I}{\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p)\frac{1}{n!}\left(\frac{jp}{h}\right)^n$$

Longitudinal broad-band resonator impedance

- The developed theory will be applied to the particular case of a longitudinal **broad-band resonator**.
- The **even terms in Eq. (1)** are responsible for the synchronous phase shift and it can be shown that, in the particular case of a broad-band resonator, their contribution **can be neglected**.

Expanding Eq. (1) up to the third order gives:

$$\ddot{\phi} + \Omega_0^2\left(S_1\phi + S_3\phi^3\right) + O(\phi^5) = 0$$

Approx. osc. frequency

$$\Omega_s(\hat{\phi}) = \Omega_0\sqrt{S_1}\left(1 + \frac{3S_3}{8S_1}\hat{\phi}^2\right) \quad (2)$$

Broad-band resonator:

$$Z_{\parallel}^{BB}(\omega) = \frac{R_{\parallel}}{1 - jQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

With:

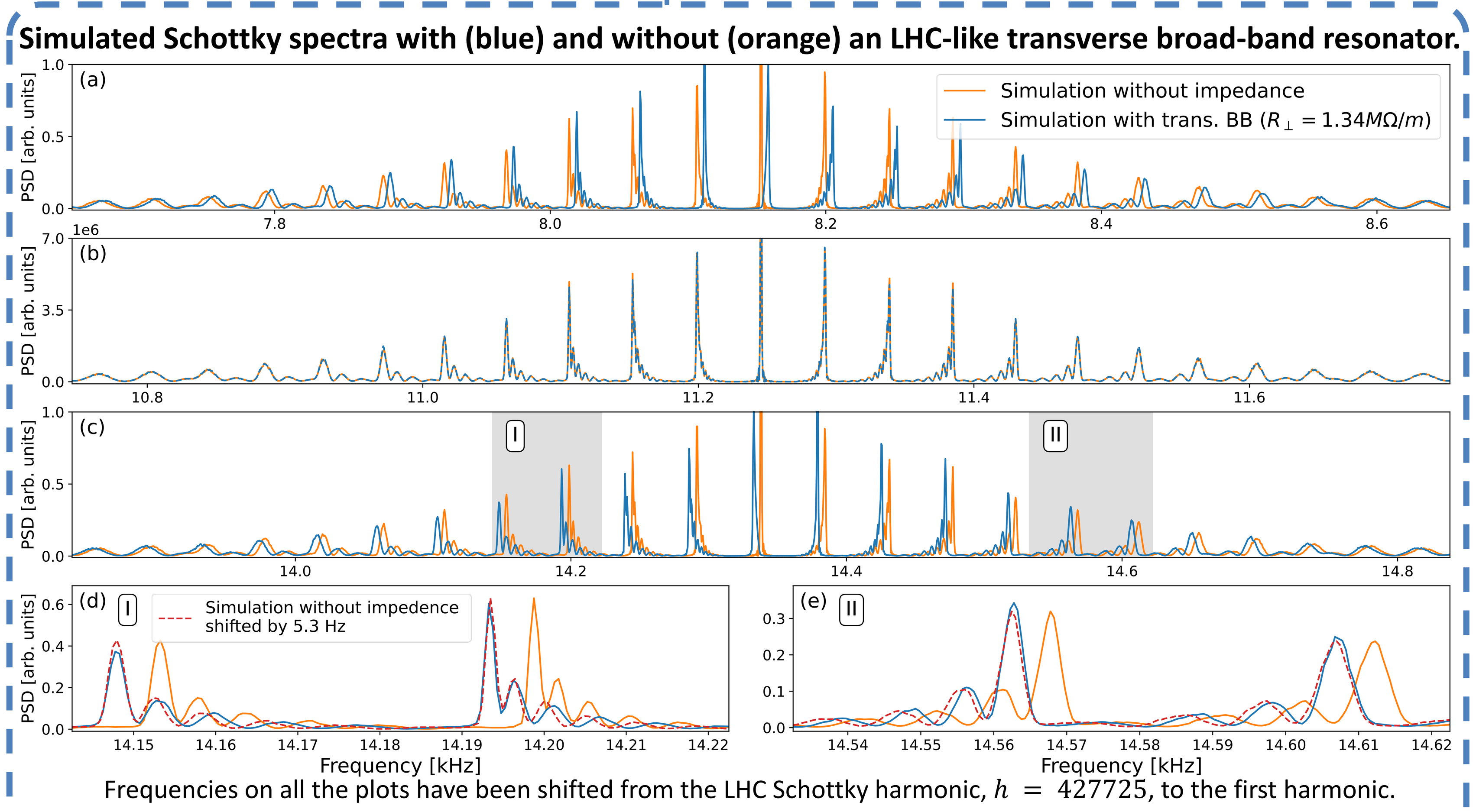
- $R_{\parallel}$: shunt impedance.
- ω_r : cut-off frequency.
- Q : quality factor.

III. Simulations

- The simulations are conducted with PyHEADTAIL [7] and aim to reproduce the typical conditions of an **LHC proton fill at injection**.
- The value of the parameters chosen for the transverse and longitudinal broad-band resonators correspond to a **significant part of the impedance in the LHC** that can be modelled as a **broad-band** resonator.

Transverse broad-band resonator impedance

Simulated Schottky spectra with (blue) and without (orange) an LHC-like transverse broad-band resonator.



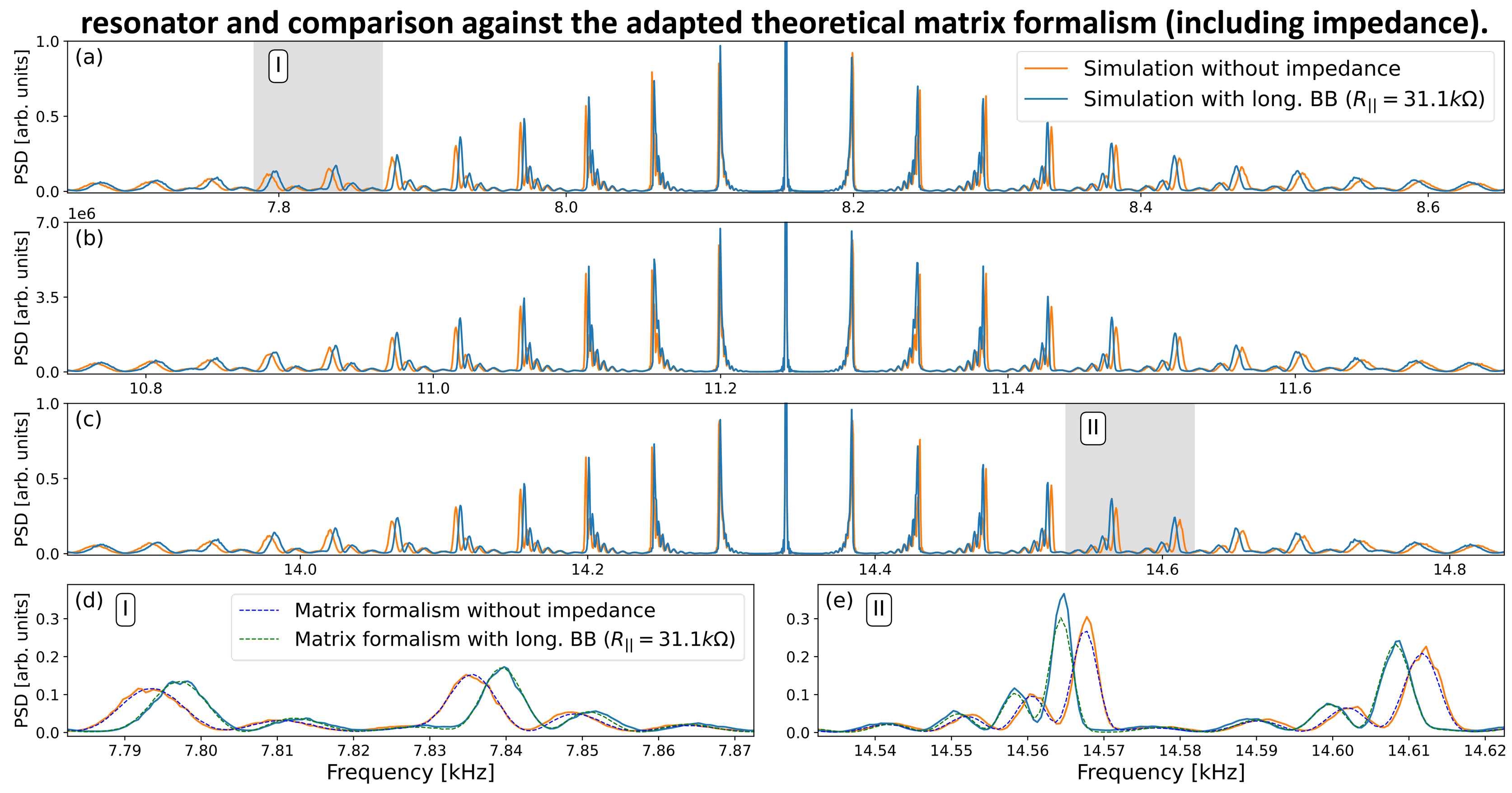
Frequencies on all the plots have been shifted from the LHC Schottky harmonic, $h = 427725$, to the first harmonic.

The following effects of a **transverse** broad-band resonator can be observed:

- The longitudinal band (b) is not affected by the transverse impedance.
- A **betatron tune shift** is visible on the transverse bands (a and c) (all satellites in a given transverse sideband are displaced by about 5Hz in the same direction). The direction of the satellite's shift - toward the right (resp. left) for the lower (resp. upper) sideband – indicates that the broad-band resonator decreases the betatron tune.
- The satellites are not simply shifted but their **internal structure is also affected by impedance**, as visible from the red dashed line in Fig. (d) and (e).

Longitudinal broad-band resonator impedance

Simulated Schottky spectra with (blue) and without (orange) an LHC-like longitudinal broad-band resonator and comparison against the adapted theoretical matrix formalism (including impedance).

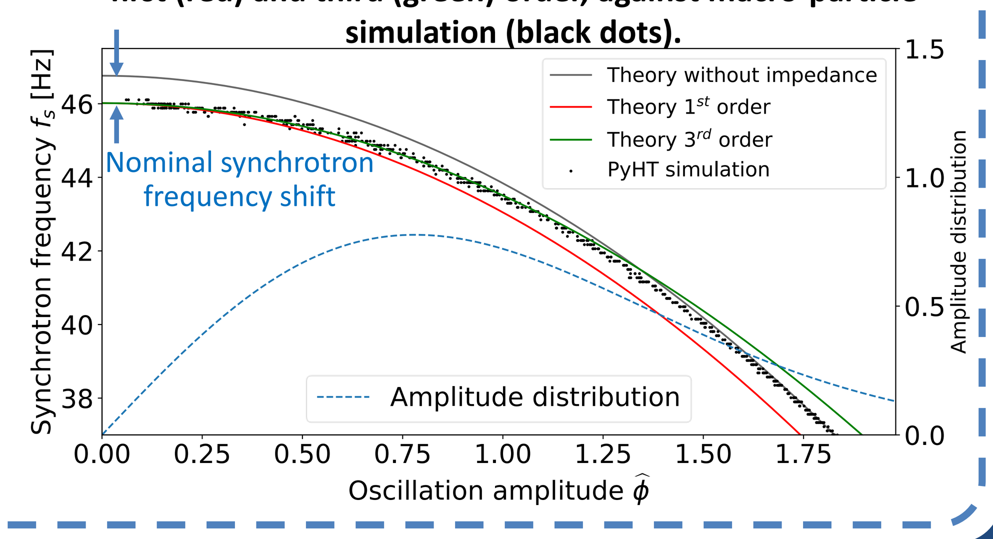


The macro-particle simulation is compared against the theoretical matrix formalism, where the relation between synchrotron amplitudes and frequencies has been generalized with Eq. (2) to include impedance effects. A good agreement is obtained between the theory and the simulation.

The following effects of the **longitudinal** broad-band resonator can be observed:

- **Shift of the nominal synchrotron frequency.** All the satellites converge toward the central one. This shift is due to the term S_1 in Eq. (2) and the new nominal synchrotron frequency is $\Omega_0\sqrt{S_1}$.
- The broad-band resonator will reduce the nominal synchrotron frequency for a machine operating above transition.
- **Amplitude dependent synchrotron frequency shift** due to the higher order terms S_{2n+1}, $n \geq 1$.

Comparison of Eq. (2) including impedance terms Z_n up to the first (red) and third (green) order, against macro-particle simulation (black dots).



V. Conclusion

- The aim of this study was to explore the effects of impedance on the Schottky spectrum.
- The **longitudinal** equation of motion was generalized to include the forces coming from **impedance**, allowing existing theoretical reconstruction methods of Schottky spectra to include impedance effects.
- The developed **theory was shown to be in good agreement with macro-particle simulations**, by correctly reproducing the amplitude dependent synchrotron tune shift, and the sub-structure of the Schottky spectrum satellites.
- The effect of **transverse impedance** on the Schottky spectra was also studied through simulation. A transverse broad-band resonator causes a betatron tune shift and affect the internal structure of the transverse satellites.

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